An Evaluation of the Decision-Making Process in Online Algorithms: A Case Study of the Standard Secretary Problem and Its Variations

Navraj Narula
nnn2112@columbia.edu
Computer Science Department
Columbia University
New York, NY 10027 USA

Abstract

The secretary problem – sometimes labeled as the marriage problem, the sultan's dowry problem, or the googol game – is really a problem concerned with selecting the "best choice" option present in a group of n items, based on relative ranking. Its applications are widely felt in the hiring realm, and to any other context whose entire selection pool may not be known firsthand. Since the problem's emergence in the mid 20^{th} century, mathematicians and other interested researchers have extended its features in a variety of directions. This paper will discuss the secretary problem as it is presented standardly in length. It will also consider the postdoc variant of the problem, in which a second-best candidate is chosen in place of the best candidate, and the double-choice variant of the problem, in which either the best or second-best candidate can be chosen for a position that allows multiple hiring choices. Results indicate the highest probability of success when considering the latter variant: the ability to choose more than one candidate for a position.

1. Introduction

An online algorithm is an algorithm that preprocesses its inputs in a piece-by-piece fashion in the order that the input is fed into the algorithm. Insertion sort is a fine example of this, since it does not require the entire input to be made available prior to running the algorithm. As the online

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algorithm progresses, a decision must be made on each input. This type of immediate feedback may encourage the algorithm to perform suboptimally compared to one which is aware of its entire input. If we are still concerned with sorting, selection sort fits this latter category; that is, it is an offline algorithm.

The online algorithm of concern in this paper is the secretary problem. As noted by Babioff et al., "allocation of resources under uncertainty" is a tangible problem felt in many real-life scenarios [1]. For instance, employers must decide whether or not to accept or reject a candidate for a job position without knowing the possible potential of future candidates who apply. The secretary problem may also be applicable to situations involving scheduling, auctioning, or even dating. Like the secretary problem, all such related problems employ the theory of optimal stopping – which is defined by choosing a time to take a particular action in order to both maximize a reward and minimize the cost. The question of interest here is: How well do online algorithms – specifically the secretary problem and its variants – approximate to the optimal solution, despite not being presented the complete input?

The discussion of the following sections leading up to the Conclusion will present a definition of each secretarial problem of interest and analyze the optimality of each one in efforts to answer the posed question.

2. The Standard Secretary Problem

2.1. Problem Definition

As alluded to earlier, the secretary problem has been extended in many different directions; so much so, that is now a field of interest within mathematics and optimization. Ferguson defines the secretary problem in its base form to have the following features [2]:

1. The number of positions available for the secretary position is one.
2. The number of applicants for the secretary position is unknown, or rather it is n.
3. The n number of applicants are interviewed sequentially in a random order. Any other ordering is equally likely.
4. A decision must be made to accept or reject each applicant.
5. The aforementioned decision must be based only on the "relative ranks" [2] of applicants interviewed so far.

6. An applicant who has previously been rejected cannot be called back for another interview.
7. The employer is satisfied with "nothing but the very best" [2] candidate. If the best candidate is hired, the payoff is 1 (success). Otherwise, it is 0 (failure).

2.2. Problem Analysis

We can see that the standard solution for the secretary problem follows a clear and concise procedure. Specifically, we can show that for some integer $r \geq 1$, $r-1$ candidates for the position are rejected. The next applicant is chosen among the best among the relative ranking of previously seen applicants. The probability, $\phi(r)$, of selecting the best applicant is then $1/n$ for r = 1. For $r > 1$, Ferguson provides the mathematical definition below [2]. (Note that $P(j)$ is the probability that the j^{th} applicant is the best candidate and also the selected candidate for the position):

$$\phi(r) = \sum_{j=r}^{n} P(j)$$

$$= \sum_{j=r}^{n} (\frac{1}{n})(\frac{r-1}{j-1}) = (\frac{r-1}{n}) \sum_{j=r}^{n} \frac{1}{j-1}$$

The value that maximizes $P(j)$, or the probability previously described, is the optimal r. According to Ferguson, the optimal r is easily computed for small values of n. If n tends to infinity (that is, n is a large value), we can then introduce a variable m to be the limit of r/n and introduce the variable k for j/n and dk for $1/n$ [2]. Ferguson states that this sum translates to a "Riemann approximation to an integral," made clearer in a modification of his definition below [2]:

$$\phi(r) = (\frac{r-1}{n}) \sum_{j=r}^{n} (\frac{n}{j-1})(\frac{1}{n})$$

$$\rightarrow \int_{m}^{1} (\frac{1}{k}) dk = -m \log(m)$$

Once the value m is solved for by derivative setting, we find that $m = 1/\epsilon$. This is roughly equivalent to 37 percent - which indicates that the optimal probability derived for the standard secretary problem is $1/\epsilon$.

From the results presented by Ferguson [2], along with support of related results in works by Gilbert and Mosteller [3], it would be "approximately" optimal for employers to wait until about 37 of the applicants have been interviewed to select the "relatively" best one. The success ratio for the standard solution is also around 37 percent.

3. Postdoc Variant

3.1. Problem Definition

The ultimate goal of the standard secretary problem is to hire nothing but the best candidate among a group of n applicants for a position. As previously outlined, the probability of success for the standard problem is $1/\epsilon$, or rather 37 percent. Vanderbei considers a slight variation of the problem. In this variation, the employer is not aiming to pick the best candidate for the position, but rather the *second* best candidate for the position. The motivation for this problem is that the best candidate for a postdoc position will receive and accept an offer from Harvard – or more generally, accept an offer at a more well-renowned company than the one he or she is interviewing for [4]. In this case, it might be more viable for the employer to focus on the second best candidate to avoid failure in acquiring the correct candidate for the position.

The optimal strategy in this case would be to reject the first half of applicants for the position, and then accept the first "second-best-so-far" applicant that arrives after rejecting the first set [4].

3.2. Problem Analysis

Vanderbei seeks to model the postdoc variant of the secretary problem as a sequential decision problem, borrowing from Dynkin in regards to his paper concerning optimal choice [5]. Proceeding in this manner, we can start our analysis with $k = 0, 1 \ldots n$. The value v_k will then represent the probability of success using the optimal strategy – assuming that k candidates have already been interviewed and none of them have been hired. Like the standard secretary problem, the decision of whether or not to hire the candidate is made immediately. After a decision is made – whether that may be a "yes" or "no" decision – the employer will continue to interview remaining candidates to see if the "correct" choice has been made [5]. Since the employer is interviewing all candidates and is thus able to judge each value in the sequence, this problem may be likened to an offline algorithm.

After interviewing k candidates, Vanderbei assumes that one of those k candidates has been selected. This candidate is the second-best among the k candidates interviewed so far. [5]. We can then let c_k indicate the probability that the currently selected second-best candidate will still be the second-best candidate after interviewing all n candidates. The formula for c_k is given below for two cases:

$$\text{For k} = n, c_k = 1$$

$$\text{For } 2 \leq k < n, c_k = \frac{k-1}{k+1} c_{k+1}$$

When the $k+1^{st}$ applicant is interviewed, the applicant will be either the first or second best among the $k+1^{st}$ candidates seen so far. Vanderbei then introduces f_k to indicate the probability that the hired applicant will be ranked second best after interviewing all other applicants [5]. As in the previous case, it is helpful to consider the next candidate – who is either going to be the best candidate so far or not. The former case would "drop" the hired candidate to second place, while the latter case would not change the position of the currently hired candidate. Vanderbei expresses f_k as:

$$\text{For k} = n, f_k = 0$$

$$\text{For } 1 \leq k < n, f_k = \frac{k}{k+1} f_{k+1} + \frac{1}{k+1} c_{k+1}$$

Vanderbei now considers the value function, v_k. If we suppose that we have interviewed k applicants and have rejected them all, the value function, v_k, $k = 0, 1 \ldots n$, indicates the probability of eventually hiring someone who will "eventually" turn out to be the second best candidate. There are three cases to consider regarding the $k+1^{st}$ candidate when k applicants have been interviewed, but none of the candidates have been hired so far. The cases are summarized from Vanderbei's paper below [5]:

1. The probability that the candidate would be worse than the best and second best candidate seen so far is $\frac{k-1}{k+1}$. Therefore, there is no reason to hire this candidate.
2. The probability of hiring a candidate that is better than all k candidates seen so far is f_{k+1}. Therefore, if we pass on hiring the candidate, the probability of success would be v_{k+1}. We would pick to hire the

candidate associated with the value with the highest probability of success in this case.

3. The probability of hiring a candidate that is the second best of all k candidates seen so far is c_{k+1}. Therefore, if we pass on hiring the candidate, the probability of success would again be v_{k+1}. We would pick to hire the candidate associated with the value with the highest probability of success in this case.

These statements are made clear in the following Hamilton-Jacobi-Bellman equation for v_k derived by Vanderbei [5]:

$$\text{For k} = n, v_k = 0$$

$$\text{For } 1 \leq k < n, f_k = \frac{k-1}{k+1} v_{k+1} + \frac{1}{k+1} \max\,(v_{k+1}, f_{k+1}) + \frac{1}{k+1} \max\,(v_{k+1}, c_{k+1})$$

$$\text{For k} = 0,\ \max\,(v_{k+1}, f_{k+1})$$

Vanderbei then goes on to define k_0 as:

$$\min = \{\, k \ \mid\ 2k \geq n-1 \,\}$$

Upon defining various theorems related to the above value, v_k, k_0, and previous calculations as visible in his paper, Vanderbei concludes by stating the optimal strategy for selecting the second-best candidate: "Reject the first k_0 applicants. After that hire the first second-best-so-far applicant that comes along" [5]. By employing this optimal strategy known as the postdoc variant, the probability of hiring the second-best applicant is:

$$v_0 = \frac{k_0(n-k_0)}{n(n+1)} \approx \frac{1}{4}$$

Compared to the secretary problem, the problem of finding the second-best candidate from a pool of n applicants for a postdoc (or secretarial) position has a more explicit solution. In summary, the optimal strategy for this solution to to accept the $first$ second-best-so-far candidate that arrives after the first half-set of rejected applicants. The probability of success, as demonstrated earlier, is the about $\frac{1}{4}$, or 25 percent. Interestingly, it is harder to select the second-best candidate for a position than it is to select the first-best candidate – whose demonstrated probability of success mirrors that of the standard solution, which is 37 percent.

4. Double Choice Variant

4.1. Problem Definition

Another variation of the secretary problem is brought to light by Freeman in his paper on variations and extensions of the standard secretary problem [6]. In the standard version of the problem, the employer seeks to hire only one candidate for the position. However, what might the case look like in terms of optimality if the employer is able to hire more than one candidate for a certain position? Tamaki extensively explored this possibility in his earlier paper concerning the secretary problem with "double choices" [7]. The optimal strategy for this variant would be to choose either the best or the second best applicant from a group of n applicants for a position.

4.2. Problem Analysis

Tamaki employed a dynamic programming approach to solving the problem. In order to illustrate the process, we consider Sakaguchi's dynamic programming equation [8] in which he defines the state, (r, s), to indicate the r^{th} applicant that is currently observed and s number of choices an employer has to make regarding each candidate. States (∞, s) indicate that the n^{th}, or last, applicant has already been observed and is not a candidate for the position. States $(\infty, 0)$ indicate that a decision has been made on all k choices. If the employer accepts a candidate, the transition probabilities from state (r, s) are $\frac{r}{j(j-1)}$ to state $(j, s-1)$ and $\frac{r}{n}$ to state $(\infty, s-1)$. If an employer rejects a candidate, the same probabilities will result in a transition to state (j, s) to (∞, s). Sakaguchi's equation is made more explicit below (Note: V denotes value), courtesy of Freeman [6]:

$$V(r,s) = \max(\frac{r}{n} + \sum_{j=r+1}^{n} \frac{r}{j(j-1)} V(j, s-1), \sum_{j=r+1}^{n} \frac{r}{j(j-1)} V(j,s))$$

Referring to the equation described above, Tamaki indicates that since the employer has two choices to make in hiring the best candidate, the employer should accept the r^{th} applicant for the position. Relative ranks would then shift to $s = 1$ for the best candidate and $s = 2$ for the second-best candidate, where $r \geq r_1^*$ and $r \geq r_2^*$.

From Tamaki and Sakaguchi's analysis of the problem [7,8], Freeman concludes the probability of success for the secretary problem when accepting more than one candidate – specifically the best or second best candidate - to be exactly 0.7934 percent [6]. This occurs when limiting values as n approaches ∞ are given. When considering the secretary problem under the double choice variant, the probability of success is much higher than the standard solution (37 percent) and certainly much higher than the postdoc variant (25 percent).

5. Conclusion

Among the variations of the standard secretary problem discussed in this paper, the probability of success for selecting more than one candidate – namely, two – for a position exceeds the probability of success for selecting only the second-best candidate for a position. As a conjecture, this statement makes sense, but mathematically proving it truly sheds light on the hiring process. It is also valuable to realize that in selection, focusing on the best candidate rather than the second-best candidate is a priority. By doing so, the probability of success increases by over 10 percent. This paper concludes that the standard secretary problem proves to be the optimal "variant" when selecting only one candidate for one available position. Other variations and extensions to the secretary problem also exist. In future work, it may be intriguing to explore such extensions related to ranking or recall in perhaps an auctioning setting, or a realm that has been less discussed than hiring.

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